If the function

$$\lambda(\zeta) = \left(\frac{p}{\pi}\right)^2 (\omega^2\mu_0\epsilon - \gamma^2) \quad (6)$$

is expansible in an absolutely convergent Fourier cosine series [5]

$$\lambda(\zeta) = \theta_0^{(h)} + 2 \sum_{n=1}^{\infty} \theta_n^{(h)} \cos 2n\zeta \quad (7)$$

(5) becomes

$$\frac{d^2 f^{(h)}}{d\zeta^2} + \left(\theta_0^{(h)} + 2 \sum_{n=1}^{\infty} \theta_n^{(h)} \cos 2n\zeta\right) f^{(h)} = 0 \quad (8)$$

which is the canonical form of Hill's equation. The computation of stability charts for Hill's equation, and, thus, the determination of the pass band and stop band structure of the dispersion characteristics, follows from the characteristic equation for Hill's equation

$$\sin^2 \frac{\pi\beta}{2} = \Delta^{(h)}(0) \sin^2 \frac{\pi\sqrt{\theta_0^{(h)}}}{2} \quad (9)$$

where β denotes the propagation factor in the Floquet solution to Hill's equation and $\Delta^{(h)}(0)$ is an infinite determinant whose elements are

$$\Delta^{(h)}(0)\,|_{mm} = 1 \quad (10)$$

$$\Delta^{(h)}(0)\,|_{mn} = \frac{\theta^{(h)}m - n}{\theta_0^{(h)} - 4m^2} \quad (m \neq n). \quad (11)$$

This procedure has been described in [1] and [3]. The solution to Hill's equation may be obtained, when β is known, by means of a procedure outlined in [5] and [6]

In the case of TM wave propagation, one introduces into (2) the substitutions

$$\zeta = \frac{\pi z}{p} \quad (12)$$

$$U^{(e)}(z) = \epsilon^{1/2} f^{(e)}(\zeta) \quad (13)$$

yielding the differential equation for $f^{(e)}(\zeta)$,

$$\frac{d^2 f^{(e)}}{d\zeta^2} + \left[\frac{1}{2\epsilon}\frac{d^2\epsilon}{d\zeta^2} - \frac{3}{4\epsilon^2}\left(\frac{d\epsilon}{d\zeta}\right)^2 + \left(\frac{p}{\pi}\right)^2 (\omega^2\mu_0\epsilon - \gamma^2)\right] f^{(e)} = 0. \quad (14)$$

If ϵ is an even-periodic function, so also is the function in square brackets and, thus, one may write

$$[\quad] = \theta_0^{(e)} + 2 \sum_{n=1}^{\infty} \theta_n^{(e)} \cos 2n\zeta \quad (15)$$

if the series is absolutely convergent. Hence,

$$\frac{d^2 f^{(e)}}{d\zeta^2} + \left[\theta_0^{(e)} + 2 \sum_{n=1}^{\infty} \theta_n^{(e)} \cos 2n\zeta\right] f^{(e)} = 0 \quad (16)$$

which is again the canonical form of Hill's equation.

Thus, the z dependence of both TE and TM waves in periodic media is expressible in terms of Hill functions. The pass band and stop band characteristics or ω-β diagrams may be determined from the characteristic equation by numerical or graphical methods and the functional dependence of the fields from the solutions to the Hill equation.

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Note on the Measurement of Material Properties by the Strip-Line Cavity

It has been found that when making measurements of the properties of materials with a strip-line cavity [1]–[3], results are obtained which are consistently lower than expected. The error is the more serious, the higher the value of the dielectric constant or permeability, as the case may be, of the sample.

In the case of measurements of magnetic properties, the reason for the effect has been explained elsewhere [4]. The discrepancy is attributable to demagnetizing factors in the specimen, and when this is in the form of a flat slab, placed either vertically against the end wall of the cavity or horizontally on the strip, the true relative permeability of an isotropic specimen $\bar{\mu}$ is given by

$$\bar{\mu} = \frac{\mu(1 - N)}{1 - \mu N} \quad (1)$$

where μ is the apparent permeability given by the perturbation formulae of [1] and [3], and N is the demagnetizing factor of the specimen appropriate to the direction of the microwave magnetic field (in MKS units). It is apparent from this formula that the difference between μ and $\bar{\mu}$ increases with increasing μ and $\bar{\mu}$, being zero for $\mu = \bar{\mu} = 1$. For large values of $\bar{\mu}$, μ approaches the limiting value $1/N$.

In the case of measurements of the dielectric constant, the discrepancy is attributable to the presence of minute air gaps between the specimen and the strip and ground plane. The perturbation formula of [1] and [3] gives a value for the dielectric constant ϵ which would be correct if the sample fitted flush with the strip and ground plane. If there is an appreciable gap, the value of ϵ calculated from the perturbation formula is an apparent one. The relation between ϵ and the true dielectric constant $\bar{\epsilon}$ can be calculated by means of the concept of a dielectric circuit analogous to the well-known magnetic circuit. The result is

$$\bar{\epsilon} = \frac{\epsilon(1 - x/h)}{1 - \epsilon x/h} \quad (2)$$

where x is the total gap height, i.e., the sum of the gaps at top and bottom of the sample, and h is the distance from the strip to the ground plane, i.e., the distance $b-t$ in the notation of [1]–[3]. It is apparent from this formula that the difference between ϵ and $\bar{\epsilon}$ increases with increasing ϵ and $\bar{\epsilon}$, being zero for $\epsilon = \bar{\epsilon} = 1$. For large values of $\bar{\epsilon}$, ϵ approaches the limiting value h/x.

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On Mode Losses in Confocal Resonator and Transmission Systems

In a recent correspondence Lonngren and Beyer [1] calculated the losses for a single iteration in a "beam waveguide" [2] with circular lenses separated by twice their focal length. Since the confocal Fabry-Perot resonator with two identical circular mirrors may be studied by superimposing two guided wave beams propagating oppositely in the given system, the beam-waveguide losses allow one to determine the resonator Q. The problem which Lonngren and Beyer [1] have solved approximately is to find the eigenvalues $\gamma_{\alpha,n}(c)$ of the integral equation

$$\gamma_{\alpha,n}(c)\mathcal{S}_{\alpha,n}(c, x) = \int_0^1 cJ_\alpha(cxy)\mathcal{S}_{\alpha,n}(c, y)ydy \quad (1)$$

for small c. In an earlier work Beyer and Scheibe [3] obtained values of $\gamma_{\alpha,n}(c)$ for large c. The purpose of this correspondence is to point out that the same information has been obtained by directly studying the solutions of (1).

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This analytic investigation (see Slepian [4] and Heurtley [5]) involves the derivation of a differential equation (using the method of commuting operators) for the functions $\mathcal{S}_{\alpha,n}(c, x)$,[1] viz.,

$$(1 - x^2)\mathcal{S}_{\alpha,n}''(c, x) + \left(\frac{1}{x} - 3x\right)\mathcal{S}_{\alpha,n}'(c, x) + \left(\Gamma_{\alpha,n}(c) - \frac{3}{4} - c^2x^2 - \frac{\alpha^2}{x^2}\right)\mathcal{S}_{\alpha,n}(c, x) = 0. \quad (2)$$

Here the differential equation eigenvalue $\Gamma_{\alpha,n}(c)$ is determined by requiring that $\mathcal{S}_{\alpha,n}$ be finite for $x=0$, 1. Using the solutions of the differential equation Slepian [4] obtains the following results for $\gamma_{\alpha,n}(c)$.

1) Fixed α, n. Small c.

$$\gamma_{\alpha,n}(c) = \frac{(-1)^n\Gamma(n+1)\Gamma(n+\alpha+1)c^{2n+\alpha+1}}{2^{2n+\alpha+1}\Gamma(2n+\alpha+1)\Gamma(2n+\alpha+2)} \cdot \left[1 - \frac{(2n+\alpha+1)\alpha^2c^2}{4(2n+\alpha)^2(2n+\alpha+2)^2} + 0(c^4)\right]^2. \quad (3)$$

2) Fixed α, n. Asymptotically large c.

$$\gamma_{\alpha,n}(c) = (-1)^n\left\{1 - \frac{\pi 2^{2\alpha+4n+2}c^{2n+\alpha+1}e^{-2c}}{\Gamma(n+1)\Gamma(n+\alpha+1)}[1 + 0(c^{-1})]\right\}. \quad (4)$$

He has also considered the case of fixed α and asymptotically large n and c. In (3) and (4) the parameter c is given by

$$c = \frac{kR^2}{2z_0}, \quad (5)$$

where k is the wavenumber, R is the radius of the circular reflector or lens, and $2z_0$ is equal to the reflector or lens separation; z_0 is their focal length. The power loss in decibel per iteration is given by

$$20 \log_{10} [\,|\gamma_{\alpha,n}(c)|\,]; \quad (6)$$

in [1] and [3] the authors used the parameter a instead given by

$$a = \left(\frac{k}{2z_0}\right)^{1/2} R = c^{1/2}.$$

Losses calculated using (3) and (4) agree well with those given in references 1 and 3, as well as McCumber's [6] recent tabulations.

Similar results for rectangular mirrors or lenses have also been given by Slepian [7].

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[1] The $S_{\alpha,n}$ are called "generalized prolate Spheroidal functions" by Slepian and "hyperspheroidal functions" by Heurtley.
[2] Note the error in equation (97) of Slepian [4]; the values listed in Table I of [4] are correct.

Ferromagnetic Resonance Linewidth and *g*-Factor in Ferrites from 2 to 18 Gc/s

The parameters of the uniform precession in homogeneously biased ferrite spheres such as the linewidth and the *g*-factor are often used to characterize ferrite materials. As is well known these data are not always frequency independent. When they are measured using the standard cavity technique it is very difficult to obtain information over a broad frequency band. On the other hand, it is the actual frequency dependence that may be of interest both for microwave applications and for a theoretical understanding of the loss mechanism in ferrites.

The purpose of this correspondence is to present room-temperature measurements on spheres of some polycrystalline ferrites and yttrium iron garnet (YIG) obtained by the crossguide coupler technique proposed by Stinson [1]. *X*-band and Ku-band waveguide couplers with standard cross sections were used. The diameter of the coupling holes were 4 mm and 3 mm, respectively. For the lower band from 2 to 8.2 Gc/s a coaxial coupler [2] was constructed from a (3.0 mm/6.5 mm) coaxial line with a coupling hole of 4-mm diameter. The wall thickness of the coupling hole was in all cases about 0.3 mm. The ferrite spheres with a diameter of 1 mm throughout all measurements were fixed in the guides by polyfoam slabs.

In contrast to Stinson's original arrangement also, the secondary guide was short-circuited at one port in a distance of a half-guide wavelength from the coupling hole. Hence, the power coupled to the matched detector is increased by 6 dB in contrast to the case when the two arms of the secondary guide are matched. This may be important for measurements on broad linewidth ferrites of small volume. In addition, we have calculated the influence of the radiation damping of the primary and the secondary waveguide provided the coupler is excited by a matched source. The power coupling from the primary to the secondary waveguide with a short located a multiple of a half-waveguide wavelength from the center of the coupling hole can than be written as

$$\frac{C}{dB} = 20 \log \left| K_1 \frac{4\pi^2r^3}{3ab\lambda_g} \right| + 20 \log \left| \frac{\chi_{xy}}{1 + K_2 j \dfrac{4\pi^2r^3}{3ab\lambda_g}\chi_{xx}} \right| \quad (1)$$

where a and b are the waveguide wide and narrow dimensions, respectively, r is the radius of the ferrite sphere, λ_g the guide wavelength, and χ_{xx} and χ_{xy} are the diagonal and off-diagonal magnetic susceptibility elements, respectively, defined in terms of the external microwave magnetic field. The quantities K_1 and K_2 depend on the circuitry of the secondary guide. When the two secondary arms are matched $K_1=1$ and $K_2=1.5$; and when one arm is matched and the other arm shorted at $\lambda_g/2$ from the hole, $K_1=2$ and $K_2=2$. If the radiation damping is not regarded, $K_2=0$ and we obtain Stinson's formula with $K_1=1$. The calculation is based on the assumption that the diameter of the sphere is much smaller than the diameter of the hole and that the wall thickness is much smaller than the diameter of the sphere. The ferrite linewidth ΔH_m measured at constant frequency and obtained from the difference of the two dc magnetic field strengths at the 3-dB points then follows from (1) as

$$\Delta H_m = \Delta H + K_2 \frac{4\pi^2r^3}{3ab\lambda_g} M_s. \quad (2)$$

Here ΔH is the linewidth defined as the difference $H_2 - H_1$ for

$$|\chi_{xy}(H_{1,2})|^2 = 0.5\,|\chi_{xy}|^2_{\max},$$

and M_s is the saturation magnetization of the ferrite. The influence of the radiation damping can be neglected at broad linewidth materials and $r<1$ mm. With single crystal garnets at $\Delta H \approx 0.5$ A/cm, however, the error can become large in the order of one-hundred per cent at usual dimensions and frequencies [3]. An additional shift of the resonance by the radiation damping however can be neglected in all these cases.

The frequency dependence of the linewidth ΔH obtained in this manner for some polycrystalline materials (R1, R5, and R6 from General Ceramics, YIG from Microwave Chemicals Lab., both U.S.A., and FXC4B and FXC5E1 from Philips, Germany) is shown in Fig. 1.

The behavior of the YIG linewidth is characterized by the sharp peak at 4 Gc/s where the uniform precession enters the spinwave manifold, in accordance with measurements first reported by Buffler [4]. The very high linewidth values of R1, FXC5E1, and FXC4B at low frequencies can be understood for these ferrites are no longer saturated at the corresponding resonance fields. Besides, it is remarkable that for all materials the linewidth of the uniform precession inside the spinwave manifold increases more or less with increasing frequencies as can be expected from theory.

When the resonance frequency of the uniform precession in a sphere is written in the

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